

NON-LOCAL BOUNDARY CONDITIONS IN EUCLIDEAN QUANTUM GRAVITY

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Abstract. Non-local boundary conditions for Euclidean quantum gravity are proposed, consisting of an integro-differential boundary operator acting on metric perturbations. In this case, the operator P on metric perturbations is of Laplace type, subject to non-local boundary conditions; by contrast, its adjoint is the sum of a Laplacian and of a singular Green operator, subject to local boundary conditions. Self-adjointness of the boundary value problem is correctly formulated by looking at Dirichlet-type and Neumann-type realizations of the operator P , following recent results in the literature. The set of non-local boundary conditions for perturbative modes of the gravitational field is written in general form on the Euclidean 4-ball. For a particular choice of the non-local boundary operator, explicit formulae for the boundary value problem are obtained in terms of a finite number of unknown functions, but subject to some consistency conditions. Among the related issues, the problem arises of whether non-local symmetries exist in Euclidean quantum gravity.

1. Introduction

The last decade of efforts on the problem of boundary conditions in (one-loop) Euclidean quantum gravity has focused on a *local formulation*, by trying to satisfy the following requirements:

- (i) Local nature of the boundary operators [1–5].
- (ii) Operator on metric perturbations, say P , and ghost operator, say Q , of Laplace type [5].
- (iii) Symmetry, and, possibly, (essential) self-adjointness of the differential operators P and Q [4, 5].
- (iv) Strong ellipticity of the boundary value problems obtained from the operators P and Q , with local boundary operators B_1 and B_2 , respectively [5, 6].
- (v) Gauge- and BRST-invariance of the boundary conditions and/or of the out-in (one-loop) amplitude [4–8].

At about the same time, in the applications to quantum field theory and quantum gravity, non-local boundary conditions had been studied mainly for operators of Dirac type (see, however, [9]), relying on the early work by Atiyah, Patodi and Singer on spectral asymmetry and Riemannian geometry [10]. What is non-local, within that framework, is the separation of the spectrum of a first-order elliptic operator (the Dirac operator on the boundary) into its positive and negative parts. This leads, in turn, to an unambiguous identification of positive- and negative-frequency modes of the (massive or massless) Dirac field, and half of them are set to zero on the bounding surface [8, 11–13].

On the other hand, non-local boundary conditions for operators of Laplace type had already been studied quite intensively in the literature, from at least two points of view:

- (i) The rich mathematical theory of pseudo-differential boundary value problems, where both the differential operator P and the boundary operator B may be replaced by integro-differential operators [14]. One may consider, for example, the boundary value problem

on a bounded open set $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$:

$$Pu = f \quad \text{in } \Omega \quad (1.1)$$

$$Tu = \varphi \quad \text{at } \partial\Omega \quad (1.2)$$

where P is the Laplace operator $-\Delta$ (from the point of view of the leading symbol, it is more convenient to define the Laplace operator with a negative sign in front of all second derivatives) and T is a *trace operator*, e.g. $Tu = \gamma_0 u \equiv [u]_{\partial\Omega}$ in the Dirichlet case, or $Tu = \gamma_1 u \equiv [u;_N]_{\partial\Omega}$ in the Neumann case, where $[_N]$ denotes the inward-pointing normal derivative at $\partial\Omega$. The theory we are interested in includes both the system

$$\mathcal{A} \equiv \begin{pmatrix} P \\ T \end{pmatrix} : C^\infty(\overline{\Omega}) \rightarrow C^\infty(\overline{\Omega}) \times C^\infty(\partial\Omega) \quad (1.3)$$

and its solution operator, or *parametrix*,

$$\mathcal{A}^{-1} \equiv \begin{pmatrix} R & K \end{pmatrix} : C^\infty(\overline{\Omega}) \times C^\infty(\partial\Omega) \rightarrow C^\infty(\overline{\Omega}). \quad (1.4)$$

With this notation, R is the *Green operator* [14] solving the problem (cf (1.2))

$$Pu = f \quad \text{in } \Omega \quad \text{and} \quad Tu = 0 \quad \text{at } \partial\Omega \quad (1.5)$$

and K is the *Poisson operator* solving the problem (cf (1.1))

$$Pu = 0 \quad \text{in } \Omega \quad \text{and} \quad Tu = \varphi \quad \text{at } \partial\Omega. \quad (1.6)$$

The Green operator can be expressed in more detail as

$$R = Q_\Omega + G \quad (1.7)$$

where Q_Ω is the pseudo-differential operator (see appendix A) defined by

$$Qf \equiv C_n \int \frac{f(x)}{|x - y|^{n-2}} dx \quad (1.8)$$

truncated to Ω (this implies extending f by 0 on \mathbb{R}^n/Ω , applying Q , and restricting to Ω), and G is a special term, called a *singular Green operator*, adapted to the choice of boundary conditions. Thus, to get a general calculus, one has to consider systems of the form [14]

$$\mathcal{A} \equiv \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} \quad (1.9)$$

for some integers j, k, j', k' such that

$$\mathcal{A} : C^\infty(\overline{\Omega})^j \times C^\infty(\partial\Omega)^k \rightarrow C^\infty(\overline{\Omega})^{j'} \times C^\infty(\partial\Omega)^{k'} \quad (1.10)$$

where P is a pseudo-differential operator on \mathbb{R}^n , P_+ is its truncation to Ω , G is a singular Green operator acting in Ω , T is a trace operator $T : \Omega \rightarrow \partial\Omega$, K is a Poisson operator $K : \partial\Omega \rightarrow \Omega$, and S is a pseudo-differential operator acting on the boundary of Ω .

The trace operators we are interested in can take the form [14]

$$T_0 u \equiv \gamma_0 u + T'_0 u \quad (1.11)$$

or, instead [14],

$$T_1 u \equiv \gamma_1 u + S_0 \gamma_0 u + T'_1 u. \quad (1.12)$$

With this notation, one has [14]

$$\gamma_j u \equiv \left[(;\cdot)_N^j u \right]_{\partial\Omega} \quad j = 0, 1, \dots \quad (1.13)$$

Moreover, T'_0 and T'_1 are integral operators going from Ω to $\partial\Omega$, and the map S_0 acts on functions on $\partial\Omega$. For example, in population theory, one studies the condition [14]

$$u(0) = \int_0^\infty u(t) f(t) dt \quad (1.14)$$

expressing the number $u(0)$ of newborn individuals as a function of the age profile $u(t)$.

This is a special case of the homogeneous condition, with $\Omega = \mathbb{R}_+$ (cf (1.11))

$$\gamma_0 u + T'_0 u = 0 \quad (1.15)$$

with [14]

$$-T'_0 u = \int_0^\infty u(t) f(t) dt. \quad (1.16)$$

(ii) Bose–Einstein condensation models, where integro-differential boundary operators lead to the existence of *bulk states* and *surface states* [15]. More precisely, given the function $q \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, one defines [15]

$$q_R(x) \equiv \frac{1}{2\pi R} \sum_{l=-\infty}^{\infty} e^{ilx/R} \int_{-\infty}^{\infty} e^{-ily/R} q(y) dy. \quad (1.17)$$

The function q_R is, by construction, periodic with period $2\pi R$, and tends to q as R tends to ∞ . On considering the region

$$B_R \equiv \{x, y : x^2 + y^2 \leq R^2\} \quad (1.18)$$

one studies the Laplacian acting on square-integrable functions on B_R , with non-local boundary conditions given by [15]

$$\left[u; N \right]_{\partial B_R} + \oint_{\partial B_R} q_R(s - s') u(R \cos(s'/R), R \sin(s'/R)) ds' = 0. \quad (1.19)$$

In polar coordinates, the resulting boundary value problem reads [15]

$$-\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} \right) = Eu \quad (1.20)$$

$$\frac{\partial u}{\partial r}(R, \varphi) + R \int_{-\pi}^{\pi} q_R(R(\varphi - \theta)) u(R, \theta) d\theta = 0. \quad (1.21)$$

For example, when the eigenvalue E is positive in equation (1.20), the corresponding eigenfunction reads [15]

$$u_{l,E}(r, \varphi) = J_l(r\sqrt{E}) e^{il\varphi} \quad (1.22)$$

where J_l is the standard notation for the Bessel function of first kind of order $l \in \mathbb{Z}$. On denoting by \tilde{q} the Fourier transform of q , and inserting (1.22) into the boundary

condition (1.21), one finds an equation leading, *implicitly*, to the knowledge of the positive eigenvalues, i.e.

$$\left[\sqrt{E} J'_l(R\sqrt{E}) + J_l(R\sqrt{E}) \tilde{q}(l/R) \right] = 0. \quad (1.23)$$

The solutions which decay rapidly away from the boundary are the surface states, whereas the solutions which remain non-negligible are called bulk states [15].

In the analysis of pseudo-differential boundary value problems, one studies the heat equation for the operator B :

$$\left(\frac{\partial}{\partial t} + B \right) u(t) = 0 \quad \text{for } t > 0 \quad (1.24)$$

with initial condition

$$u(0) = u_0 \quad (1.25)$$

where B is an operator acting like $P_+ + G$ and with a domain defined by the boundary condition $Tu = 0$. Such an operator is called a *realization* of P . One of the aims of functional calculus is to make sense of the exponentiation e^{-tB} under suitable assumptions on B . More precisely, one wants to investigate e^{-tB} so as to get detailed information both on the *solutions* in terms of their data and on the *kernel* of the solution operator and its trace [14]. For this purpose, a basic tool is the analysis of the resolvent

$$R_\lambda \equiv (B - \lambda I)^{-1} \quad (1.26)$$

which makes it possible to study also other functions of B , defined by the Cauchy integral formula

$$f(B) = \frac{i}{2\pi} \int_\gamma f(\lambda) R_\lambda d\lambda \quad (1.27)$$

with γ a curve in the complex plane going around the spectrum of B .

So far, we have tried to convince the general reader that there are many good reasons for studying pseudo-differential boundary value problems with their functional calculus on the one hand, and their applications to Euclidean quantum gravity on the other hand. Now we can outline the plan of our paper, which is as follows. Section 2, relying on [14],

describes how to build the adjoint of a Laplacian, when non-local boundary conditions are imposed. This scheme is then applied to the gravitational field in section 3, with a particular choice of integro-differential boundary conditions. A mode-by-mode form of such boundary conditions is studied in section 4, when the background consists of the Euclidean 4-ball. Concluding remarks are presented in section 5, and relevant details are given in the appendices.

2. The adjoint with non-local boundary conditions

In the case of the gravitational field, inspired by section 1, we consider a scheme where the differential operator on metric perturbations remains of Laplace type (as well as the ghost operator), whereas the boundary conditions are of integro-differential nature. This means that the full boundary operator, say $B_{ab}{}^{cd}$, may be expressed as the sum of a local operator, say $\tilde{B}_{ab}{}^{cd}$, obtained from projectors and first-order differential operators [4, 5], and an integral operator, so that the boundary conditions read (see appendix B)

$$\begin{aligned} \left[B_{ab}{}^{cd} h_{cd}(x) \right]_{\partial M} &= \left[\tilde{B}_{ab}{}^{cd} h_{cd}(x) \right]_{\partial M} \\ &+ \left[\int_M W_{ab}{}^{cd}(x, x') h_{cd}(x') dV' \right]_{\partial M} \end{aligned} \quad (2.1a)$$

where dV' denotes the integration measure over M . We may now decide, following DeWitt [16], that unprimed lower-case indices refer to the point x and primed lower-case indices refer to the point x' . This leads to

$$\left[B_{ab}{}^{cd} h_{cd}(x) \right]_{\partial M} = \left[\tilde{B}_{ab}{}^{cd} h_{cd}(x) \right]_{\partial M} + \left[\int_M W_{ab}{}^{c'd'} h_{c'd'} dV' \right]_{\partial M} \quad (2.1b)$$

which is the form of the boundary conditions chosen hereafter.

Since we are concerned, for simplicity, with operators of Laplace type in a flat four-dimensional background (all curvature effects result then from the boundary only), it is very important for us to understand the effect of integro-differential boundary conditions on such a class of operators, motivated by section 1 and bearing in mind Eq. (2.1b). For

this purpose, following [14], we remark that, after integration by parts, one finds the Green formula for $P = -\Delta$, $u \in D(P)$, and v in the domain $D(P^*)$ of the adjoint of P :

$$(Pu, v)_\Omega = (-\Delta u, v)_\Omega = (u, -\Delta v)_\Omega + (\mathcal{U}\rho u, \rho v)_{\partial\Omega} \quad (2.2)$$

where the *Green matrix* [14] reads, in our case,

$$\mathcal{U} = i \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad (2.3)$$

whilst ρ is the (Cauchy) boundary operator, whose action reduces to

$$\rho u = (\gamma_0 u, \gamma_1 u). \quad (2.4)$$

The same property (2.4) holds for $v \in D(P^*)$. Suppose now that the boundary conditions are expressed in the integro-differential form (cf (1.11))

$$[\gamma_0 u + T'_0 u]_{\partial\Omega} = 0.$$

The term $(\mathcal{U}\rho u, \rho v)_\Gamma$ in Eq. (2.2), which is equal to

$$(\mathcal{U}\rho u, \rho v)_{\partial\Omega} = i(\gamma_1 u, \gamma_0 v)_{\partial\Omega} + i(\gamma_0 u, \gamma_1 v)_{\partial\Omega} \quad (2.5a)$$

can be then re-expressed as

$$(\mathcal{U}\rho u, \rho v)_{\partial\Omega} = i(\gamma_1 u, \gamma_0 v)_{\partial\Omega} + i(-T'_0 u, \gamma_1 v)_{\partial\Omega} \quad (2.5b)$$

which implies that P^* , the (formal) adjoint of P , can be obtained by adding to $-\Delta$ a singular Green operator, i.e. [14]

$$P^* v = -\Delta v + iT_0'^* \gamma_1 v \quad (2.6)$$

supplemented by the local boundary condition

$$\gamma_0 v = 0 \text{ at } \partial\Omega. \quad (2.7)$$

By contrast, if the boundary conditions (cf (1.12))

$$\left[\gamma_1 u + S_0 \gamma_0 u + T_1' u \right]_{\partial \Omega} = 0$$

are imposed, which modify the standard Neumann case, it is convenient to re-express $\gamma_1 u$, at the boundary, in the form

$$\gamma_1 u = -S_0 \gamma_0 u - T_1' u \quad (2.8)$$

and insert Eq. (2.8) into Eq. (2.5a). This implies that the adjoint of P now reads

$$P^* v = -\Delta v + iT_1'^* \gamma_0 v \quad (2.9)$$

subject to the local boundary condition

$$\gamma_1 v = 0 \text{ at } \partial \Omega. \quad (2.10)$$

In other words, we are discovering a property which is known to some mathematicians, but not so familiar to physicists: if an elliptic differential operator (here taken to be of Laplace type) is studied with integro-differential boundary conditions, its adjoint is a pseudo-differential operator, subject to local boundary conditions.

Self-adjointness problems are properly formulated by studying the realization of the operator P [14]. In our case, this means adding to the Laplacian a singular Green operator, and considering a trace operator which expresses the integro-differential boundary conditions. More precisely, a Dirichlet-type realization of $P = -\Delta$ is the operator [14]

$$B_D \equiv \left(-\Delta + G_D \right)_{T_0} \quad (2.11)$$

where (see (1.11))

$$G_D \equiv K_1 \gamma_1 + G' \quad (2.12)$$

$$T_0 \equiv \gamma_0 + T_0'. \quad (2.13)$$

In our paper, the Poisson operators K_i , for $i = 0, 1$, are completely determined by the requirement of self-adjointness. Indeed, the domains of B_D and its adjoint coincide [14] if and only if (cf (2.6) and (2.12))

$$K_1 = iT_0'^* \quad (2.14)$$

$$G' = G'^* . \quad (2.15)$$

Moreover, a Neumann-type realization of $P = -\Delta$ is the operator

$$B_N \equiv \left(-\Delta + G_N \right)_{T_1} \quad (2.16)$$

where (see (1.12))

$$G_N \equiv K_0 \gamma_0 + F' \quad (2.17)$$

$$T_1 \equiv \gamma_1 + S_0 \gamma_0 + T'_1 . \quad (2.18)$$

Following [14], we use the notation (2.11) and (2.16) for particular realizations of the Laplacian, but a notation along the lines of (1.9) if we want to stress the properties of a system in the functional calculus of the boundary value problem. The domains of B_N and its adjoint are found to coincide [14] if and only if (cf (2.9) and (2.17))

$$K_0 = i T_1'^* \quad (2.19)$$

$$S_0 = -S_0^* \quad (2.20)$$

$$F' = F'^* . \quad (2.21)$$

3. Application to the gravitational field

In the case of the gravitational field, our boundary operator (2.1b) corresponds to the trace operator (2.13). The local boundary operator $\tilde{B}_{ab}{}^{cd}$ is taken to be the one for which the following conditions are imposed on metric perturbations on a 3-sphere boundary of radius a [2]:

$$\left[h_{ij} \right]_{\partial M} = 0 \quad (3.1)$$

$$\left[h_{0i} \right]_{\partial M} = 0 \quad (3.2)$$

$$\left[\frac{\partial h_{00}}{\partial \tau} + \frac{6}{\tau} h_{00} - \frac{\partial}{\partial \tau} (g^{ij} h_{ij}) \right]_{\partial M} = 0 \quad (3.3)$$

where $\tau \in [0, a]$. Equations (3.1)–(3.3) express, to our knowledge, the only set of local boundary conditions which are of Dirichlet type on h_{ij} and h_{0i} , and for which strong ellipticity of the boundary value problem is not violated [5, 6]. Since we only want to modify the Dirichlet sector of such boundary conditions, which is expressed by (3.1) and (3.2), we have to require that (see (2.1b))

$$W_{00}{}^{c'd'} = 0 \quad \forall c', d'. \quad (3.4)$$

Thus, we eventually consider the system (cf our Eq. (1.9), and Eq. (1.6.84) in [14])

$$\mathcal{A} \equiv \begin{pmatrix} -\Delta + G \\ T \end{pmatrix} \quad (3.5)$$

where T is of the type (2.13) in its ij and $0i$ components, i.e.

$$\left[T_{ij}{}^{cd} h_{cd}(x) \right]_{\partial M} = \left[h_{ij}(x) \right]_{\partial M} + \left[\int_M W_{ij}{}^{c'd'} h_{c'd'} dV' \right]_{\partial M} \quad (3.6)$$

$$\left[T_{0i}{}^{cd} h_{cd}(x) \right]_{\partial M} = \left[h_{0i} \right]_{\partial M} + \left[\int_M W_{0i}{}^{c'd'} h_{c'd'} dV' \right]_{\partial M} \quad (3.7)$$

and of the type (3.3) (cf Eq. (2.18) and set $T'_1 = 0$ therein) in its normal component h_{00} , i.e.

$$\left[T_{00}{}^{cd} h_{cd}(x) \right]_{\partial M} = \left[\frac{\partial h_{00}}{\partial \tau} + \frac{6}{\tau} h_{00} - \frac{\partial}{\partial \tau} (g^{ij} h_{ij}) \right]_{\partial M}. \quad (3.8)$$

Moreover, $-\Delta$ is the standard Laplacian on metric perturbations in flat Euclidean 4-space, and G may be viewed as the direct sum of G_D and G_N (cf example 1.6.16 in [14]), with

$$G_D = K_1 \gamma_1 + G' \quad (3.9)$$

$$G_N = F' \quad (3.10)$$

subject to the self-adjointness conditions

$$\left[(K_1)_{jl}{}^{cd} h_{cd} \right]_{\partial M} = i \left[\int_M W_{jl}{}^{c'd'} h_{c'd'} dV' \right]_{\partial M}^* \quad (3.11)$$

$$\left[(K_1)_{0j}{}^{cd}h_{cd}\right]_{\partial M} = i \left[\int_M W_{0j}{}^{c'd'} h_{c'd'} dV'\right]_{\partial M}^* \quad (3.12)$$

$$\left[(K_1)_{00}{}^{cd}h_{cd}\right]_{\partial M} = 0 \quad (3.13)$$

$$G'_{ab}{}^{cd} = \left(G'_{ab}{}^{cd}\right)^* \quad (3.14)$$

$$F'_{ab}{}^{cd} = \left(F'_{ab}{}^{cd}\right)^* . \quad (3.15)$$

Note that, by virtue of Eq. (3.4), the counterpart of T'_1 (see (2.18)) vanishes in our problem (as we said after (3.7)), and hence K_0 vanishes as well (see (2.19)), so that G_N reduces to F' as we write in (3.10). Furthermore, the condition (2.20) is satisfied by virtue of the boundary condition (3.8).

4. Mode-by-mode equations on the Euclidean 4-ball

At this stage it can be helpful to write down a set of equations for perturbative modes of the gravitational field, once that the right-hand sides of (3.6) and (3.7), and the left-hand side of (3.3), are set to zero at the boundary. As in section 3, our background is a portion of flat Euclidean 4-space bounded by a 3-sphere (hereafter, \vec{x} corresponds to local coordinates on the 3-sphere; \vec{x} and τ , altogether, correspond to the symbol x used before). This is quite important in (one-loop) quantum cosmology [7, 8, 11–13], and also as a first step towards more complicated field-theoretical models. Under the above assumptions, the expansion of metric perturbations on a family of 3-spheres centred on the origin reads [8]

$$h_{00}(\vec{x}, \tau) = \sum_{n=1}^{\infty} a_n(\tau) Q^{(n)}(\vec{x}) \quad (4.1)$$

$$h_{0i}(\vec{x}, \tau) = \sum_{n=2}^{\infty} \left[b_n(\tau) \frac{Q^{(n)}(\vec{x})}{(n^2 - 1)} + c_n(\tau) S_i^{(n)}(\vec{x}) \right] \quad (4.2)$$

$$\begin{aligned}
h_{ij}(\vec{x}, \tau) &= \sum_{n=3}^{\infty} u_n(\tau) \left[\frac{Q_{|ij}^{(n)}(\vec{x})}{(n^2 - 1)} + \frac{1}{3} c_{ij} Q^{(n)}(\vec{x}) \right] \\
&+ \sum_{n=1}^{\infty} \frac{1}{3} e_n(\tau) c_{ij} Q^{(n)}(\vec{x}) \\
&+ \sum_{n=3}^{\infty} \left[f_n(\tau) \left(S_{i|j}^{(n)}(\vec{x}) + S_{j|i}^{(n)}(\vec{x}) \right) + z_n(\tau) G_{ij}^{(n)}(\vec{x}) \right]. \tag{4.3}
\end{aligned}$$

Here, with a standard notation, $Q^{(n)}(\vec{x})$, $S_i^{(n)}(\vec{x})$ and $G_{ij}^{(n)}(\vec{x})$ are the scalar, transverse vector, transverse-traceless tensor hyperspherical harmonics on a unit 3-sphere (with metric c_{ij}), respectively. On denoting by μ a parameter with dimension $[\text{length}]^{-1}$, by I_r the modified Bessel function of first kind and order r , and on choosing the de Donder gauge-averaging functional, one then finds for the transverse-traceless modes the formula [8]

$$z_n(\tau) = \alpha_n \tau I_n(\mu\tau) \tag{4.4}$$

whilst the vector modes read [8]

$$c_2(\tau) = \varepsilon I_3(\mu\tau) \tag{4.5}$$

$$c_n(\tau) = \tilde{\varepsilon}_{1,n} I_{n+1}(\mu\tau) + \tilde{\varepsilon}_{2,n} I_{n-1}(\mu\tau) \tag{4.6}$$

$$f_n(\tau) = \tau \left[-\frac{\tilde{\varepsilon}_{1,n}}{(n+2)} I_{n+1}(\mu\tau) + \frac{\tilde{\varepsilon}_{2,n}}{(n-2)} I_{n-1}(\mu\tau) \right] \tag{4.7}$$

and the scalar modes are given by [8]

$$a_1(\tau) = \frac{1}{\tau} \left[A_1 I_1(\mu\tau) + A_4 I_3(\mu\tau) \right] \tag{4.8}$$

$$e_1(\tau) = \tau \left[3A_1 I_1(\mu\tau) - A_4 I_3(\mu\tau) \right] \tag{4.9}$$

$$a_2(\tau) = \frac{1}{\tau} \left[B_1 I_2(\mu\tau) + B_4 I_4(\mu\tau) \right] \tag{4.10}$$

$$b_2(\tau) = B_2 I_2(\mu\tau) - B_4 I_4(\mu\tau) \tag{4.11}$$

$$e_2(\tau) = \tau \left[3B_1 I_2(\mu\tau) - 2B_2 I_2(\mu\tau) \right] \tag{4.12}$$

$$a_n(\tau) = \frac{1}{\tau} \left[\rho_{1,n} I_n(\mu\tau) + \rho_{3,n} I_{n-2}(\mu\tau) + \rho_{4,n} I_{n+2}(\mu\tau) \right] \quad (4.13)$$

$$b_n(\tau) = \rho_{2,n} I_n(\mu\tau) + (n+1)\rho_{3,n} I_{n-2}(\mu\tau) - (n-1)\rho_{4,n} I_{n+2}(\mu\tau) \quad (4.14)$$

$$u_n(\tau) = \tau \left[-\rho_{2,n} I_n(\mu\tau) + \frac{(n+1)}{(n-2)} \rho_{3,n} I_{n-2}(\mu\tau) + \frac{(n-1)}{(n+2)} \rho_{4,n} I_{n+2}(\mu\tau) \right] \quad (4.15)$$

$$e_n(\tau) = \tau \left[3\rho_{1,n} I_n(\mu\tau) - 2\rho_{2,n} I_n(\mu\tau) - \rho_{3,n} I_{n-2}(\mu\tau) - \rho_{4,n} I_{n+2}(\mu\tau) \right]. \quad (4.16)$$

Now we follow the procedure outlined in the introduction in a simpler case (see (1.20)–(1.23)), i.e. we insert the mode solutions (4.4)–(4.16) of the eigenvalue equation for metric perturbations into the mode-by-mode form of the boundary condition resulting from (3.6), (3.7) and (3.3). For this purpose, it is convenient to define

$$\kappa_{0i} \equiv \int_M W_{0i}{}^{c'd'} h_{c'd'} dV' \quad (4.17)$$

$$\kappa_{ij} \equiv \int_M W_{ij}{}^{c'd'} h_{c'd'} dV'. \quad (4.18)$$

The boundary conditions are then expressed by (3.3) jointly with the equations

$$\left[h_{0i} + \kappa_{0i} \right]_{\partial M} = 0 \quad (4.19)$$

$$\left[h_{ij} + \kappa_{ij} \right]_{\partial M} = 0. \quad (4.20)$$

The tensor fields κ_{0i} and κ_{ij} , representing the non-local contribution to the boundary conditions for a symmetric rank-two tensor field, are themselves symmetric. They can be therefore expanded on a family of 3-spheres centred on the origin according to formulae entirely analogous to (4.2) and (4.3), with the modes $\{b_n, c_n\}$ replaced by the modes $\{\tilde{b}_n, \tilde{c}_n\}$,

say, and the modes $\{u_n, e_n, f_n, z_n\}$ replaced by the modes $\{\tilde{u}_n, \tilde{e}_n, \tilde{f}_n, \tilde{z}_n\}$. The boundary conditions (3.3), (4.19) and (4.20) lead, therefore, to the following set of equations for perturbative modes, for all $n \geq 3$:

$$\left[\frac{da_n}{d\tau} + \frac{6}{\tau} a_n - \frac{1}{\tau^2} \frac{de_n}{d\tau} \right] (\tau = a) = 0 \quad (4.21)$$

$$[b_n + \tilde{b}_n](\tau = a) = 0 \quad (4.22)$$

$$[c_n + \tilde{c}_n](\tau = a) = 0 \quad (4.23)$$

$$[u_n + \tilde{u}_n](\tau = a) = 0 \quad (4.24)$$

$$[e_n + \tilde{e}_n](\tau = a) = 0 \quad (4.25)$$

$$[f_n + \tilde{f}_n](\tau = a) = 0 \quad (4.26)$$

$$[z_n + \tilde{z}_n](\tau = a) = 0. \quad (4.27)$$

The finite-dimensional spaces corresponding to the modes (4.5) and (4.8)–(4.12) should, of course, be treated separately, by requiring that

$$\left[\frac{da_k}{d\tau} + \frac{6}{\tau} a_k - \frac{1}{\tau^2} \frac{de_k}{d\tau} \right] (\tau = a) = 0 \quad \text{if } k = 1, 2 \quad (4.28)$$

$$[b_2 + \tilde{b}_2](\tau = a) = [c_2 + \tilde{c}_2](\tau = a) = 0 \quad (4.29)$$

$$[e_2 + \tilde{e}_2](\tau = a) = 0. \quad (4.30)$$

It is now convenient to write more explicitly the tensor fields κ_{0i} and κ_{ij} , by using the definitions (4.17) and (4.18) on the one hand, and the expansions analogous to (4.2) and (4.3) on the other hand. This leads to

$$\begin{aligned} \kappa_{0i}(\vec{x}, \tau) &= \int_M \left[W_{0i}{}^{0'0'} h_{0'0'} + 2W_{0i}{}^{(0'k')} h_{0'k'} + W_{0i}{}^{k'l'} h_{k'l'} \right] dV' \\ &= \sum_{n=2}^{\infty} \left[\tilde{b}_n(\tau) \frac{Q_{|i}^{(n)}(\vec{x})}{(n^2 - 1)} + \tilde{c}_n(\tau) S_i^{(n)}(\vec{x}) \right] \end{aligned} \quad (4.31)$$

$$\begin{aligned}
\kappa_{ij}(\vec{x}, \tau) &= \int_M \left[W_{ij}{}^{0'0'} h_{0'0'} + 2W_{ij}{}^{(0'k')} h_{0'k'} + W_{ij}{}^{k'l'} h_{k'l'} \right] dV' \\
&= \sum_{n=3}^{\infty} \tilde{u}_n(\tau) \left[\frac{Q_{|ij}^{(n)}(\vec{x})}{(n^2-1)} + \frac{1}{3} c_{ij} Q^{(n)}(\vec{x}) \right] \\
&\quad + \sum_{n=1}^{\infty} \frac{1}{3} \tilde{e}_n(\tau) c_{ij} Q^{(n)}(\vec{x}) \\
&\quad + \sum_{n=3}^{\infty} \left[\tilde{f}_n(\tau) \left(S_{i|j}^{(n)}(\vec{x}) + S_{j|i}^{(n)}(\vec{x}) \right) + \tilde{z}_n(\tau) G_{ij}^{(n)}(\vec{x}) \right]. \tag{4.32}
\end{aligned}$$

One should now insert the expansions (4.1)–(4.3) into the integrals occurring in (4.31) and (4.32), hence reading out the modes $\tilde{b}_n, \tilde{c}_n, \tilde{u}_n, \tilde{e}_n, \tilde{f}_n, \tilde{z}_n$ for a given form of $W_{ab}{}^{c'd'}$, chosen to be compatible with (3.4), (3.11) and (3.12). Last, such a solution should be inserted into the boundary conditions (4.21)–(4.30), which would be then re-expressed, in non-local form, uniquely in terms of modes for metric perturbations, which are known from (4.4)–(4.16).

For example, *if one assumes* that $W_{ab}{}^{c'd'}$ has distributional nature, and that suitable functions $\{f_1, \dots, f_6\}$ exist (they should be such that the integrals we are going to build exist as Lebesgue or Riemann integrals) for which (it is convenient to factorize τ'^{-3} in our ansatz, because it cancels exactly a term τ'^3 from the integration measure dV')

$$W_{0i}{}^{0'0'} = \tau'^{-3} f_1(\tau, \tau') \delta(\vec{x}, \vec{x}') \hat{\nabla}_i \tag{4.33}$$

$$2W_{0i}{}^{(0'k')} = \tau'^{-3} f_2(\tau, \tau') \delta(\vec{x}, \vec{x}') \delta_i^k \tag{4.34}$$

$$W_{0i}{}^{k'l'} = \tau'^{-3} f_3(\tau, \tau') \delta(\vec{x}, \vec{x}') c^{kl} \hat{\nabla}_i \tag{4.35}$$

where $\hat{\nabla}_i$ coincides with the covariant derivative on the boundary denoted by $|_i$ so far, the method described after Eq. (4.32) leads to, for all $n \geq 2$,

$$\begin{aligned}
\frac{\tilde{b}_n(\tau)}{(n^2-1)} &= \int_0^a \left[f_1(\tau, \tau') a_n(\tau') + f_2(\tau, \tau') \frac{b_n(\tau')}{(n^2-1)} \right. \\
&\quad \left. + f_3(\tau, \tau') e_n(\tau') \right] d\tau' \tag{4.36}
\end{aligned}$$

$$\tilde{c}_n(\tau) = \int_0^a f_2(\tau, \tau') c_n(\tau') d\tau' \quad (4.37)$$

whereas, if $n = 1$, one finds

$$f_1(\tau, \tau') a_1(\tau') + f_3(\tau, \tau') e_1(\tau') = 0 \quad (4.38)$$

which implies the non-trivial property

$$\frac{f_1(\tau, \tau')}{f_3(\tau, \tau')} = -\frac{e_1(\tau')}{a_1(\tau')}. \quad (4.39)$$

Moreover, if one assumes that

$$W_{ij}{}^{0'0'} = \tau'^{-3} f_4(\tau, \tau') \delta(\vec{x}, \vec{x}') c_{ij} \quad (4.40)$$

$$2W_{ij}{}^{(0'k')} = \tau'^{-3} f_5(\tau, \tau') \delta(\vec{x}, \vec{x}') \delta_{(i}{}^k \widehat{\nabla}_{j)} \quad (4.41)$$

$$W_{ij}{}^{k'l'} = \tau'^{-3} f_6(\tau, \tau') \delta(\vec{x}, \vec{x}') \delta_{(i}{}^k \delta_{j)}{}^l \quad (4.42)$$

the same method implies, for the first few modes,

$$\tilde{e}_1(\tau) = \int_0^a \left[3f_4(\tau, \tau') a_1(\tau') + f_6(\tau, \tau') e_1(\tau') \right] d\tau' \quad (4.43)$$

$$\int_0^a f_5(\tau, \tau') b_2(\tau') d\tau' = 0 \quad (4.44)$$

$$\int_0^a f_5(\tau, \tau') c_2(\tau') d\tau' = 0 \quad (4.45)$$

and hence, up to a zero-measure set (see (4.5) and (4.11)),

$$f_5(\tau, \tau') = 0 \quad (4.46)$$

$$\tilde{e}_2(\tau) = \int_0^a \left[3f_4(\tau, \tau') a_2(\tau') + f_6(\tau, \tau') e_2(\tau') \right] d\tau' \quad (4.47)$$

and, for all $n \geq 3$,

$$\tilde{u}_n(\tau) = \int_0^a f_6(\tau, \tau') u_n(\tau') d\tau' \quad (4.48)$$

$$\tilde{e}_n(\tau) = \int_0^a \left[3f_4(\tau, \tau')a_n(\tau') + f_6(\tau, \tau')e_n(\tau') \right] d\tau' \quad (4.49)$$

$$\tilde{f}_n(\tau) = \int_0^a f_6(\tau, \tau')f_n(\tau')d\tau' \quad (4.50)$$

$$\tilde{z}_n(\tau) = \int_0^a f_6(\tau, \tau')z_n(\tau')d\tau'. \quad (4.51)$$

The formulae (4.36), (4.37) and (4.48)–(4.51) for the coupled modes should be inserted into the boundary conditions (4.22)–(4.27), hence leading to the non-local form of the boundary conditions resulting from the assumptions (4.33)–(4.35) and (4.40)–(4.42). Nothing more explicit can be said unless one determines the form of *all* functions $\{f_1, \dots, f_6\}$.

5. Concluding remarks

Motivated by the recent developments in Euclidean quantum gravity on manifolds with boundary [1–9] and Bose–Einstein condensation models on the one hand [15], and by the progress in the functional calculus of pseudo-differential boundary value problems on the other hand [14], we have considered the mixed boundary conditions (3.3), (4.19) and (4.20) for the quantized gravitational field on the Euclidean 4-ball. The main drawback of such boundary conditions is, possibly, the apparent lack of an invariance principle leading to their derivation. On the other hand, it has been recently proved in [5, 6] that precisely the boundary conditions which are completely invariant under infinitesimal diffeomorphisms on metric perturbations lead to serious technical problems. In other words, on choosing the operator P on metric perturbations to be of Laplace type, the resulting boundary value problem fails to be strongly elliptic [5, 6]. This is a technical condition which requires that a unique solution should exist of the eigenvalue equation for the leading symbol of P , subject to a decay condition at infinite geodesic distance from the boundary and to the boundary conditions of the problem [5, 6]. If there is lack of strong ellipticity, the fibre trace of the heat-kernel diagonal acquires a non-integrable part near the boundary [5], and hence not even the one-loop semiclassical approximation is well defined (except, possibly, if one works with a “smeared” form [13] of the heat kernel for the operator P).

In the light of the above difficulties, the possible lack of an invariance principle does not seem a sufficient reason for not considering the many interesting motivations leading to the boundary conditions (3.3), (4.19) and (4.20). In particular, the Eqs. (4.4)–(4.16), jointly with (4.21)–(4.32), supplemented by the self-adjointness conditions (3.11) and (3.12), lead to what seems to be a very interesting calculational scheme for quantum gravity on the Euclidean 4-ball (see (4.33)–(4.51)). At least six outstanding problems are now in sight:

- (i) Is there a set of non-local symmetries in Euclidean quantum gravity leading to our non-local set of boundary conditions (3.3), (4.19) and (4.20) or to a suitable modification of our scheme?
- (ii) Is the resulting class of boundary value problems compatible with strong ellipticity?
- (iii) Can one build explicitly a class of bulk and surface states in Euclidean quantum gravity with non-local boundary conditions, inspired by the work in [15]? The idea is then to prove that, upon inserting (4.4)–(4.16) into (4.21)–(4.32), solutions formally analogous to the bulk and surface states of [15] do actually exist (cf (1.20)–(1.23)).
- (iv) Can one make assumptions for $W_{ab}{}^{c'd'}$ in the boundary conditions (2.1b) which admit (4.33)–(4.35) and (4.40)–(4.42) as a particular case? This would elucidate the general structure of the kernel of the trace operator in the system (3.5).
- (v) Can one study heat-kernel asymptotics with non-local boundary conditions for the gravitational field?
- (vi) Can one perform a path-integral quantization, if the non-local choice (2.1b) is made for the boundary data? What form of boundary conditions should be imposed on ghost fields?

If one were able to solve all such problems, an entirely new vision would emerge in Euclidean quantum gravity, with a non-trivial impact also on the other branches of Euclidean field theories. Thus, encouraging evidence exists that the Euclidean approach

continues to play a key role on the way towards further progress in the theory of the quantized gravitational field [8, 17].

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Appendix A

To be self-contained, it is important to describe the main ideas behind the definition of pseudo-differential operators. For this purpose, following [14], let us recall that the action of a differential operator of order m on \mathfrak{R}^n ,

$$A(x, D_x) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha \quad (A.1)$$

can be expressed, with the help of Fourier transform, as

$$A(x, D_x)u(x) = (2\pi)^{-n} \int e^{ix \cdot \xi} a(x, \xi) \hat{u}(\xi) d\xi \quad (A.2)$$

where $a : (x, \xi) \rightarrow a(x, \xi)$ is the function called *symbol* or characteristic polynomial, whose action is defined by

$$a(x, \xi) \equiv \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha. \quad (A.3)$$

Pseudo-differential operators are obtained by considering, instead of a symbol of polynomial nature as in (A.3), a more general function. More precisely, a pseudo-differential operator P with symbol $p(x, \xi)$ is the operator defined by [14]

$$(Pu)(x) \equiv (2\pi)^{-n} \int_{\mathfrak{R}^{2n}} e^{i(x-y) \cdot \xi} p(x, \xi) u(y) dy d\xi. \quad (A.4)$$

Strictly, the definition (A.4) holds for $u \in \mathcal{S}(\mathfrak{R}^n)$, but can be extended to a more general class of functions, provided that the symbol $p(x, \xi)$ satisfies suitable conditions. In

particular, it is important to consider symbols which, for some $\delta \in \mathfrak{R}$, are C^∞ functions satisfying the inequality

$$|D_x^\beta D_\xi^\alpha p(x, \xi)| \leq C_{\alpha, \beta}(x) (1 + |\xi|^2)^{\frac{\delta}{2} - \frac{|\alpha|}{2}} \quad \forall \alpha, \beta \quad (\text{A.5})$$

where $C_{\alpha, \beta}$ is a continuous function. In several applications, one needs also the asymptotic expansion of the symbol. For this purpose, one can assume that $p(x, \xi)$ is *polyhomogeneous*, i.e. it satisfies (A.5) *and* has an asymptotic expansion [14]

$$p(x, \xi) \sim \sum_{l \in \mathcal{N}} p_{\delta-l}(x, \xi) \quad (\text{A.6})$$

where each $p_{\delta-l}$ is a C^∞ function homogeneous of degree $\delta - l$ in ξ for $|\xi| > 1$, and

$$p - \sum_{l < j} p_{\delta-l}$$

is a C^∞ function satisfying the inequality (A.5) with δ replaced by $\delta - j$, for all $j \in \mathcal{N}$.

Appendix B

The construction of the local part of the boundary operator in Eq. (2.1b) deserves further comments. In other words, a rigorous formulation of local boundary conditions for operators of Laplace type involves vector bundles over M and its boundary with their sections, a matrix consisting of projectors and first-order differential operators, and the boundary data. A concise description is as follows. We consider an m -dimensional Riemannian manifold, say (M, g) , a vector bundle V over M , with a connection ∇ , and operators of Laplace type, i.e.

$$P \equiv -g^{ab} \nabla_a \nabla_b - E \quad (\text{B.1})$$

with E an endomorphism of V . The operator P maps smooth sections of V , say φ , into smooth sections of V . In the case of local boundary conditions for P , their general form is [5]

$$\begin{pmatrix} \Pi & 0 \\ \Lambda & \mathbb{I} - \Pi \end{pmatrix} \begin{pmatrix} [\varphi]_{\partial M} \\ [\varphi; N]_{\partial M} \end{pmatrix} = 0 \quad (\text{B.2})$$

where Π is a self-adjoint projection operator, and Λ is a tangential differential operator on the boundary of M :

$$\Lambda \equiv (\mathbb{I} - \Pi) \left[\frac{1}{2} \left(\Gamma^i \widehat{\nabla}_i + \widehat{\nabla}_i \Gamma^i \right) + \Sigma \right] (\mathbb{I} - \Pi). \quad (B.3)$$

With the notation used in (B.3), $\widehat{\nabla}$ is the induced connection on ∂M , Γ^i are endomorphism-valued vector fields on the boundary, and Σ is an endomorphism of the vector bundle over ∂M which is a copy of $[V]_{\partial M}$, with sections given by $[\varphi]_{\partial M}$. Γ^i and Σ are anti-self-adjoint and self-adjoint, respectively, and are annihilated by Π on the left and on the right, i.e.

$$\Pi \Gamma^i = \Gamma^i \Pi = 0 \quad (B.4)$$

$$\Pi \Sigma = \Sigma \Pi = 0. \quad (B.5)$$

As is shown in [1, 4, 5, 7, 8], one arrives at such boundary conditions whenever one tries to obtain gauge- and BRST-invariant boundary conditions in quantum field theory.

In our paper, however, to avoid losing strong ellipticity of the boundary value problem in Euclidean quantum gravity [5, 6], we always assume that

$$\Gamma^i = 0 \quad (B.6)$$

for all $i = 1, \dots, m - 1$.

References

- [1] Barvinsky A O 1987 *Phys. Lett.* **195B** 344
- [2] Luckock H C 1991 *J. Math. Phys.* **32** 1755
- [3] Esposito G and Kamenshchik A Yu 1995 *Class. Quantum Grav.* **12** 2715
- [4] Avramidi I G, Esposito G and Kamenshchik A Yu 1996 *Class. Quantum Grav.* **13** 2361
- [5] Avramidi I G and Esposito G 1997 Gauge theories on manifolds with boundary *Preprint hep-th/9710048* (to appear in *Comm. Math. Phys.*)
- [6] Avramidi I G and Esposito G 1998 *Class. Quantum Grav.* **15** 1141

- [7] Moss I G and Silva P J 1997 *Phys. Rev.* **D 55** 1072
- [8] Esposito G, Kamenshchik A Yu and Pollifrone G 1997 *Euclidean Quantum Gravity on Manifolds with Boundary (Fundamental Theories of Physics 85)* (Dordrecht: Kluwer).
- [9] Marachevsky V N and Vassilevich D V 1996 *Class. Quantum Grav.* **13** 645
- [10] Atiyah M F, Patodi V K and Singer I M 1975 *Math. Proc. Camb. Phil. Soc.* **77** 43
- [11] D'Eath P D and Halliwell J J 1987 *Phys. Rev.* **D 35** 1100
- [12] D'Eath P D and Esposito G 1991 *Phys. Rev.* **D 44** 1713
- [13] Esposito G 1998 *Dirac Operators and Spectral Geometry (Cambridge Lecture Notes in Physics 12)* (Cambridge: Cambridge University Press)
- [14] Grubb G 1996 *Functional Calculus of Pseudo-Differential Boundary Problems (Progress in Mathematics 65)* (Boston: Birkhäuser)
- [15] Schröder M 1989 *Rep. Math. Phys.* **27** 259
- [16] DeWitt B S 1967 *Phys. Rev.* **160** 1113
- [17] Gibbons G W and Hawking S W 1993 *Euclidean Quantum Gravity* (Singapore: World Scientific)